DERIVATION OF BAC - CAB RULE
USING TERM BY TERM COMPONENT EXPANSION

Use term - by - term expansion rather than summation notation to prove the following identity :

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\,(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}\,(\mathbf{A}\cdot\mathbf{B}) \tag{1}$$

- **Solution :**

The strategy here will be to write out the left hand side and right hand sides in term by term components and show they equal. Let' s start by noting explicitly that each of the vectors can be written in component form :

$$\begin{aligned}
\mathbf{A} &= A_x\,\hat{\mathbf{x}} + A_y\,\hat{\mathbf{y}} + A_z\,\hat{\mathbf{z}}\\
\mathbf{B} &= B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}}\\
\mathbf{C} &= C_x\,\hat{\mathbf{x}} + C_y\,\hat{\mathbf{y}} + C_z\,\hat{\mathbf{z}}
\end{aligned}$$

The right hand side of (1) becomes :

$$\begin{aligned}
\mathbf{B}\,(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}\,(\mathbf{A}\cdot\mathbf{B}) &= \left(B_x\,\hat{x} + B_y\,\hat{y} + B_z\,\hat{z}\right)\left(A_x\,C_x + A_y\,C_y + A_z\,C_z\right) - \\
&\quad \left(C_x\,\hat{x} + C_y\,\hat{y} + C_z\,\hat{z}\right)\left(A_x\,B_x + A_y\,B_y + A_z\,B_z\right)\\
&= \left(B_x\,A_x\,C_x + B_x\,A_y\,C_y + B_x\,A_z\,C_z - C_x\,A_x\,B_x - C_x\,A_y\,B_y - C_x\,A_z\,B_z\right)\hat{\mathbf{x}}\\
&\quad + \left(B_y\,A_x\,C_x + B_y\,A_y\,C_y + B_y\,A_z\,C_z - C_y\,A_x\,B_x - C_y\,A_y\,B_y - C_y\,A_z\,B_z\right)\hat{\mathbf{y}}\\
&\quad + \left(B_z\,A_x\,C_x + B_z\,A_y\,C_y + B_z\,A_z\,C_z - C_z\,A_x\,B_x - C_z\,A_y\,B_y - C_z\,A_z\,B_z\right)\hat{\mathbf{z}}
\end{aligned} \tag{2}$$

All the terms in eq. (2) are scalars, so the order of multiplication is irrelevant. The terms highlighted in red will cancel, leaving us with the following expression for the right side of eq. (1) :

$$\begin{aligned}
\mathbf{B}\,(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}\,(\mathbf{A}\cdot\mathbf{B}) &= \left(B_x\,A_y\,C_y - C_x\,A_y\,B_y - C_x\,A_z\,B_z + B_x\,A_z\,C_z\right)\hat{\mathbf{x}} + \\
&\quad \left(-C_y\,A_x\,B_x + B_y\,A_x\,C_x + B_y\,A_z\,C_z - C_y\,A_z\,B_z\right)\hat{\mathbf{y}} + \\
&\quad \left(B_z\,A_x\,C_x - C_z\,A_x\,B_x - C_z\,A_y\,B_y + B_z\,A_y\,C_y\right)\hat{\mathbf{z}}
\end{aligned} \tag{3}$$

Now, let' s expand the left hand side. We have two cross products, so we should begin by computing the inner cross product :

$$\mathbf{B}\times\mathbf{C} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \left(B_y C_z - B_z C_y\right)\hat{\mathbf{x}} - \left(B_x C_z - B_z C_x\right)\hat{\mathbf{y}} + \left(B_x C_y - B_y C_x\right)\hat{\mathbf{z}}$$

Now, we use these terms in the computation of the remaining cross product of **A×(B×C):**

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_y C_z - B_z C_y & B_z C_x - B_x C_z & B_x C_y - B_y C_x \end{vmatrix} =$$

$$\begin{aligned} &\left(A_y B_x C_y - A_y B_y C_x - A_z B_z C_x + A_z B_x C_z\right)\hat{\mathbf{x}} \\ &-\left(A_x B_x C_y - A_x B_y C_x - A_z B_y C_z + A_z B_z C_y\right)\hat{\mathbf{y}} \\ &+\left(A_x B_z C_x - A_x B_x C_z - A_y B_y C_z + A_y B_z C_y\right)\hat{\mathbf{z}} \end{aligned} \quad (5)$$

A term by term comparison of eqs. (3) and (5) show they are the same, and we have proven the identity.

Manipulating vector expressions using Einstein summation notation is far superior to term by term expansion; it is elegant, compact, and easier to follow (once you are familiar with the notation). The sort of term by term expansion shown above is just painful; it requires so many individual steps that you are sure to make at least one if not several errors.