PROOF of the Root Test

Part (a)

Assume that $L < 1$. We need to show that $\sum a_n$ is absolutely convergent. To do this let's first note that because $L > 1,$ there is some number r such that $L < r < 1.$

Now, recall that $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ and since $L < r,$

there is some N such that if $n \geq N$ then $|a_n|^{\frac{1}{n}} < r \Rightarrow |a_n| < r^n.$

Now the series $\sum_{n=1}^{\infty} r^n$ is a convergent geometric series because

$0 < r < 1.$ Since $|a_n| < r^n$ for $n \geq N$ the series $\sum_{n=N}^{\infty} |a_n|$ also

converges by the Comparison Test. Since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n|$

then $\sum_{n=1}^{\infty} |a_n|$ is also convergent since adding a finite number to a

convergent sequence yields a finite sum. Therefore $\sum_{n=1}^{\infty} a_n$ is

absolutely convergent.

Part (b)

Assume that $L > 1$. We need to show that $\sum a_n$ is divergent.

Recall that $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ and since $L > 1$,

there is some N such that if $n \geq N$ then $|a_n|^{\frac{1}{n}} > 1 \Rightarrow |a_n| > 1^n = 1$.

However, if $|a_n| > 1$ for all $n \geq N$ then $\lim_{n\to\infty} |a_n| \neq 0$ which means

that $\lim_{n\to\infty} a_n \neq 0$ and by the Divergence Test $\sum a_n$ is divergent.

Part (c)

Assume the $L = 1$. This is true for the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is absolutely

convergent. It is also true for the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which is conditionally

convergent, and also true for the series $\sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent.

Therefore, when $L = 1$, the test is inconclusive.